

LEVI GEOMETRY
AND THE TANGENTIAL CAUCHY-RIEMANN EQUATIONS
ON A REAL ANALYTIC SUBMANIFOLD OF \mathbb{C}^n

BY

AL BOGGESS

ABSTRACT. The relationship between the Levi geometry of a submanifold of \mathbb{C}^n and the tangential Cauchy-Riemann equations is studied. On a real analytic codimension two submanifold of \mathbb{C}^n , we find conditions on the Levi algebra which allow us to locally solve the tangential Cauchy-Riemann equations (in most bidegrees) with kernels. Under the same conditions, we show that, locally, any CR-function is the boundary value jump of a holomorphic function defined on some suitable open set in \mathbb{C}^n . This boundary value jump result is the best possible result because we also show that there is no one-sided extension theory for such submanifolds of \mathbb{C}^n . In fact, we show that if S is a real analytic, generic, submanifold of \mathbb{C}^n (any codimension) where the excess dimension of the Levi algebra is less than the real codimension, then S is not extendible to any open set in \mathbb{C}^n .

0. Introduction. Recently, there has been increasing interest in the relationship of geometric concepts, such as the Levi form of a hypersurface and the local solvability of the tangential Cauchy-Riemann equations. For example, Henkin [H] has shown that on a strictly pseudoconvex hypersurface, one can locally solve the tangential Cauchy-Riemann equations (in most bidegrees). His approach involved explicit integral kernels. In this paper we start such a program for higher codimension. We relate such concepts as the Levi form and the Levi algebra to the local solvability of the tangential Cauchy-Riemann equations on a real analytic, codimension two submanifold of \mathbb{C}^n . The geometric conditions we impose are in terms of strict pseudoconvexity of the Levi form (Definition 2.3) and the excess dimension of the Levi algebra (Definition 2.2). We represent the local solution to the tangential Cauchy-Riemann equations in terms of explicit kernels. We show (Theorem 5.1) that on a generic real analytic, strictly pseudoconvex, codimension two submanifold S of \mathbb{C}^n with excess dimension of S equal to one, we can locally solve the tangential Cauchy-Riemann equations in bidegrees (p, q) where $1 \leq q \leq n - 3$. Although our emphasis is on codimension two, we indicate an inductive process for local solvability on higher codimension submanifolds of \mathbb{C}^n .

From the kernel approach we also get, for free, a local result (Theorem 5.15) about CR-functions on a codimension two submanifold, S , satisfying the hypothesis of Theorem 5.1. Locally, we carefully construct an open set V in \mathbb{C}^n with $\overline{V} \cap S \neq \emptyset$ such that a CR-function on $\overline{V} \cap S$ is the boundary value jump across S of a

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holomorphic function defined on V . Related to this result, we prove that there is no one-sided extension theory for CR-functions on S . In fact, we show (Theorem 5.22) that on a real analytic, generic, submanifold of \mathbf{C}^n (any codimension $\leq n$) where the excess dimension is less than the (real) codimension, there does *not* exist a fixed open set in \mathbf{C}^n to which all CR-functions extend as holomorphic functions. This means that the boundary value jump result (Theorem 5.15) is the best possible result for real analytic, codimension two submanifolds with excess dimension equal to one. In the real analytic case, Theorem 5.22 also provides a converse to the result of Hunt and Wells [HW] (see also Greenfield [Gr]) which roughly states that if S is a generic submanifold with maximal excess dimension (i.e. excess dimension = (real) codimension), then S is extendible to an open set in \mathbf{C}^n .

The operators involved in the local solvability result (Theorem 5.1) are the boundary value jumps across S of kernels which are defined on carefully chosen hypersurfaces which contain S . To find these hypersurfaces, we prove the following geometric result (Theorem 3.4), which is of independent interest: A real analytic submanifold of real dimension $2n - k$ with excess dimension equal to $k - p$, $0 \leq p \leq k$, is locally contained in the intersection of p transverse real analytic, Levi flat hypersurfaces. A uniqueness part is also proven. An analogous result, without the uniqueness part, is also proven for C^∞ submanifolds.

We have organized the paper as follows: In §2 the geometric concepts such as the Levi form and the Levi algebra are explained from a Pfaffian system (differential forms) point of view. This point of view is somewhat different from the vector field approach introduced by Greenfield [Gr] and Hill and Taiani [HT], which is more standardly used. But the roots of our approach are actually more classical in nature, going back to Cartan [C]. Whenever possible, we relate the two points of view. Our approach only requires the most elementary concepts from Pfaffian systems, which we set down in §1. §3 is devoted to our geometric result stated above. The essential tools we use are the real Frobenius theorem and Tomassini's result [T], which states that a real analytic CR-function on a real analytic submanifold is locally the restriction of a unique holomorphic function. In §4, we state some formal identities which relate a class of kernels to the tangential Cauchy-Riemann equations. Few proofs are given in this section, since this material is very thoroughly discussed in Harvey and Polking [HP]. Finally, we state and prove our local solvability result in §5.

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1. Pfaffian systems. In this section, we describe the fundamental concepts of Pfaffian systems. For more details we refer the reader to Gardner [G].

Let S be a C^∞ manifold, and let $\mathcal{O}(S)$ be the ring of smooth functions on S . We let $T^*(S)$ (resp. $T(S)$) be the cotangent bundle (resp. tangent bundle) of S . If U is an open set in S , then $\Gamma^*(U)$ (resp. $\Gamma(U)$) will denote the space of smooth sections of $T^*(S)$ (resp. $T(S)$) over U .

1.1. DEFINITION. A Pfaffian system over an open set $U \subset S$ is an $\mathcal{E}(S)$ submodule of $\Gamma^*(U)$.

Dually, we define:

1.2. DEFINITION. A vector field system over an open set $U \subset S$ is an $\mathcal{E}(S)$ submodule of $\Gamma(U)$.

Let \mathcal{G} be a Pfaffian system over an open set $U \subset S$; then we define the annihilator of \mathcal{G} , $\mathring{\mathcal{G}}$, to be the set of vector fields on U which are annihilated by \mathcal{G} , i.e.

$$\mathring{\mathcal{G}} = \{X \in \Gamma(S); (\eta, X) = 0 \text{ for each } \eta \in \mathcal{G}\}.$$

We have used the notation (η, X) to be the duality pairing between the one form η and the vector field X .

In what follows, all Pfaffian systems and vector field systems are assumed to have constant dimension. If \mathcal{G} is a Pfaffian system of dimension l and if $u_1, \dots, u_l \in \mathcal{G}$ generate \mathcal{G} over $\mathcal{E}(U)$, then we often write $\mathcal{G} = \text{sp}\{u_1, \dots, u_l\}$.

1.3. DEFINITION. Let \mathcal{G} be a Pfaffian system over an open set $U \subset S$. \mathcal{G} is said to be integrable if for each $\eta \in \mathcal{G}$ we have $d_S \eta \in \mathcal{G} \wedge \Gamma^*(U)$, where $d_S: \Lambda^p T^*(S) \rightarrow \Lambda^{p+1} T^*(S)$ is the exterior derivative operator on S .

It is often useful to understand how the concepts for Pfaffian systems translate for vector fields. For example, a Pfaffian system \mathcal{G} is integrable if and only if $\mathring{\mathcal{G}}$ is involutive, i.e. $[X_1, X_2] \in \mathring{\mathcal{G}}$ whenever $X_1, X_2 \in \mathring{\mathcal{G}}$. This follows immediately from the equation

$$(1.4) \quad (d_S \eta, X_1 \wedge X_2) = (\eta, [X_1, X_2]); \quad \eta \in \mathcal{G}; X_1, X_2 \in \mathring{\mathcal{G}}.$$

The important theorem about integrable Pfaffian systems is the following Froebnius theorem (see [F, p. 97]):

1.5. THEOREM. *Suppose S is a smooth (resp. real analytic) manifold. Suppose \mathcal{G} is a smooth (resp. real analytic) integrable Pfaffian system of (real) dimension l defined in a neighborhood of a point $x_0 \in S$. Then, locally, there exist smooth (resp. real analytic) functions $u_1, \dots, u_l: S \rightarrow \mathbf{R}$ with $u_1(x_0) = \dots = u_l(x_0) = 0$; $d_S u_1(x_0) \wedge \dots \wedge d_S u_l(x_0) \neq 0$ and*

$$\mathcal{G} = \text{sp}\{d_S u_1, \dots, d_S u_l\}.$$

Let \mathcal{G} be a Pfaffian system over an open set $U \subset S$. The exterior derivative $d_S: \mathcal{G} \rightarrow \Lambda^2 \Gamma^*(U)$ is *not* an $\mathcal{E}(U)$ module homomorphism. To remedy this we let $\delta_1: \mathcal{G} \rightarrow \Lambda^2 \Gamma^*(U) / \Gamma^*(U) \wedge \mathcal{G}$ be the following composition:

$$\mathcal{G} \xrightarrow{d_S} \Lambda^2 \Gamma^*(U) \xrightarrow{\pi} \Lambda^2 \Gamma^*(U) / \Gamma^*(U) \wedge \mathcal{G},$$

where π denotes natural projection. It is clear that δ_1 is an $\mathcal{E}(U)$ module homomorphism. The kernel of δ_1 is an $\mathcal{E}(U)$ submodule of \mathcal{G} which we denote by \mathcal{G}_1 . This submodule is often called the first derived system of \mathcal{G} . We inductively define $\delta_r: \mathcal{G}_{r-1} \rightarrow \Lambda^2 \Gamma^*(U) / \Gamma^*(U) \wedge \mathcal{G}_{r-1}$ as the composition

$$\mathcal{G}_{r-1} \xrightarrow{d_S} \Lambda^2 \Gamma^*(U) \xrightarrow{\pi_{r-1}} \Lambda^2 \Gamma^*(U) / \Gamma^*(U) \wedge \mathcal{G}_{r-1},$$

where again π_{r-1} denotes natural projection. The r th derived system $\mathcal{G}_r \subset \mathcal{G}_{r-1}$ is then defined to be the kernel of δ_r . We have the exact sequence:

$$0 \rightarrow \mathcal{G}_r \rightarrow \mathcal{G}_{r-1} \xrightarrow{\delta_r} \delta_r(\mathcal{G}_{r-1}) \rightarrow 0$$

where $\delta_r(\mathcal{G}_{r-1})$ is the image of \mathcal{G}_{r-1} in $\Lambda^2\Gamma^*(U)/\Gamma^*(U) \wedge \mathcal{G}_{r-1}$ under δ_r . Clearly, we have

$$\dots \subset \mathcal{G}_r \subset \mathcal{G}_{r-1} \subset \dots \subset \mathcal{G}_1 \subset \mathcal{G}_0 = \mathcal{G},$$

and if $\mathcal{G}_r = \mathcal{G}_{r-1}$ (or, equivalently, $\delta_r = 0$), then \mathcal{G}_r is integrable and $\mathcal{G}_p = \mathcal{G}_{r-1}$ for each $p \geq r$. There is a smallest integer N , called the derived length of \mathcal{G} , such that $\mathcal{G}_{N+p} = \mathcal{G}_N$ for each $p \geq 0$. We shall often write \mathcal{G}_{int} for \mathcal{G}_N . It is clear from the definitions that \mathcal{G}_{int} is the largest integrable system of \mathcal{G} and that

$$\mathcal{G}_{\text{int}} = \bigcap_{l=1}^{\infty} \mathcal{G}_l.$$

From the vector field point of view, we have $\mathring{\mathcal{G}} \subset \mathring{\mathcal{G}}_1 \subset \dots \subset \mathring{\mathcal{G}}_r \subset \dots$, and $\mathring{\mathcal{G}}_r$ is spanned by $\mathring{\mathcal{G}}_{r-1}$ and all elements which are r th Lie brackets of elements in $\mathring{\mathcal{G}}$. We also have

$$\mathring{\mathcal{G}}_{\text{int}} = \bigcup_{l=1}^{\infty} \mathring{\mathcal{G}}_l$$

and $\mathring{\mathcal{G}}_{\text{int}}$ is the vector field system generated (over $\mathcal{E}(U)$) by $\mathring{\mathcal{G}}$ and all Lie brackets of $\mathring{\mathcal{G}}$, i.e. $\mathring{\mathcal{G}}_{\text{int}}$ is the Lie algebra of $\mathring{\mathcal{G}}$.

2. The Levi form and Levi algebra. In this section we explain the Levi form and Levi algebra in terms of a particular Pfaffian system. This point of view is useful when the information that is desired from the Levi form is in terms of differential forms (as opposed to vector fields).

We shall make use of the standard pointwise Euclidean metric for $\Lambda^r T^*(\mathbf{R}^m)$, which we denote by $\langle \phi, \psi \rangle_x$, $\phi, \psi \in \Lambda^r T^*(\mathbf{R}^m)$. To emphasize that this is a real (vs. complex hermitian) inner product, we may write $\langle \phi, \psi \rangle_x^{\mathbf{R}}$.

Now, suppose S is a generic submanifold of \mathbf{C}^n of (real) dimension $2n - k$. Let $H^*(S) = T^*(S) \cap JT^*(S)$ be the holomorphic cotangent bundle of S , where $J: T^*(\mathbf{R}^{2n}) \rightarrow T^*(\mathbf{R}^{2n})$ is the complex structure on $T^*(\mathbf{R}^{2n})$. Since S is generic, $H^*(S)$ is a Pfaffian system of minimal dimension, i.e. the dimension of each fiber is $2(n - k)$. Let \mathcal{G} be the orthogonal complement of $H^*(S)$ in $T^*(S)$. Clearly, $T^*(S) = H^*(S) \oplus \mathcal{G}$. The Pfaffian system \mathcal{G} is of real dimension k and \mathcal{G} is sometimes called the totally real subbundle of $T^*(S)$. In fact, if S is locally defined by $S = \{x \in \mathbf{C}^n; r_1(x) = \dots = r_k(x) = 0\}$ where $r_1, \dots, r_k: \mathbf{C}^n \rightarrow \mathbf{R}$ are smooth functions with $dr_1 \wedge \dots \wedge dr_k \neq 0$ on S , then locally:

$$\begin{aligned} \mathcal{G} &= \text{sp}\{Jdr_1|_S, \dots, Jdr_k|_S\} \\ &= \text{sp}\{d^c r_1|_S, \dots, d^c r_k|_S\} \quad (d^c = J \circ d = i(\partial - \bar{\partial})). \end{aligned}$$

2.1. DEFINITION. If S is generic, then we say that S is *regular* if \mathcal{G} and each of its derived systems have constant rank on S .

We shall only consider regular submanifolds in this work. This allows us to think of \mathcal{G} and its derived systems as subbundles of $T^*(S)$.

2.2. DEFINITION. The l th Levi form of S is the map $\delta_l: \mathcal{G}_{l-1} \rightarrow \Lambda^2 \Gamma^*(S)/\mathcal{G}_{l-1} \wedge \Gamma^*(S)$, where we have used the notation from §1.

Recall that if $\delta_l = 0$ then \mathcal{G}_{l-1} is an integrable subbundle of \mathcal{G} . So the Levi forms on S measure the integrability of the subbundles of the totally real subbundle $\mathcal{G} \subset T^*(S)$.

Since $\Lambda^2 T^*(S)/\mathcal{G} \wedge T^*(S)$ can be identified with $\Lambda^2 H^*(S)$, it is clear that $\delta_1(Jdr_j|_S)$ is just the orthogonal projection of $d(Jdr_j)|_S = dd^c r_j|_S = 2i\partial\bar{\partial}r_j|_S$ onto $\Lambda^2 H^*(S) = \Lambda^2(T^*(S)/\mathcal{G})$. In general, if $\eta \in \mathcal{G}$ then $\delta_1(\eta)$ is just the orthogonal projection of $d\tilde{\eta}|_S$ onto $\Lambda^2 H^*(S)$, where $\tilde{\eta}$ is any ambient extension of η . Similarly, $\delta_l(\eta)$ is just the orthogonal projection of $d\tilde{\eta}|_S$ onto $\Lambda^2(T^*(S)/\mathcal{G}_{l-1})$.

2.3. DEFINITION. Suppose S is regular. The excess dimension of the Levi algebra of S , denoted by $\text{ex dim}(S)$, is the real dimension of the quotient $\mathcal{G}/\mathcal{G}_{\text{int}}$, where \mathcal{G}_{int} is the maximal integrable subbundle of \mathcal{G} .

From §1 it is clear that $\mathcal{G}/\mathcal{G}_{\text{int}} \simeq \mathring{\mathcal{G}}/\mathring{\mathcal{G}}_{\text{int}}$. The space $\mathring{\mathcal{G}}_{\text{int}}$ is the Levi algebra generated by $\mathring{\mathcal{G}} =$ the holomorphic tangent bundle of S . So from a vector field point of view, the quantity $\text{ex dim}(S)$ measures the dimension of the Levi algebra of S relative to the dimension of the holomorphic tangent bundle of S .

It is clear that $0 \leq \text{ex dim}(S) \leq k = \text{codim}(S)$. A submanifold S of \mathbf{C}^n is said to be Levi flat if $\text{ex dim}(S) = 0$ or, equivalently, if \mathcal{G} is integrable.

2.4. DEFINITION. Suppose S is a submanifold of \mathbf{C}^n and $x_0 \in S$. S is said to be strictly pseudoconvex at x_0 if there exists an $\eta \in \mathcal{G}$ and a constant $C > 0$ such that

$$\langle \delta_1(\eta), \omega \wedge J\omega|_{x_0} \rangle_{x_0}^{\mathbf{R}} \geq C |\omega|_{x_0}^2$$

for each $\omega \in H_{x_0}^*(S)$.

This inequality is equivalent to the following inequality:

$$\frac{i}{2} \langle d\tilde{\eta}, W \wedge \bar{W} \rangle_{x_0}^{\mathbf{C}} \geq C |W|_{x_0}^2,$$

where $W = \omega - iJ\omega$ is in the complexified holomorphic cotangent bundle, $\langle \cdot, \cdot \rangle_{x_0}^{\mathbf{C}}$ denotes the complex hermitian inner-product on forms, and $\tilde{\eta}$ is any ambient extension of η .

For example, suppose S is a real hypersurface in \mathbf{C}^n , locally defined by $S = \{x \in \mathbf{C}^n; r(x) = 0\}$ where $r: \mathbf{C}^n \rightarrow \mathbf{R}$ is smooth with $dr \neq 0$ on S . Clearly $\mathcal{G} = \text{sp}\{Jdr|_S\} = \text{sp}\{d^c r|_S\}$ and $\text{ex dim}(S) \leq 1$. S is strictly pseudoconvex at a point $x_0 \in S$ if either r or $-r$ satisfies

$$\langle \partial\bar{\partial}r, W \wedge \bar{W} \rangle_{x_0}^{\mathbf{C}} \geq C |W|^2$$

for any W which is in the complexified holomorphic cotangent space of S at x_0 . Written out in coordinates, $W = (w_1, \dots, w_n)$; this is equivalent to

$$\sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(x_0) w_j \bar{w}_k \geq C |W|^2$$

provided $\sum_{j=1}^n (\partial r / \partial z_j) w_j = 0$.

So the above concepts about the Levi form agree with the classical concepts about the Levi form on a hypersurface. Moreover, in higher codimension, the above concepts also agree with those concepts defined from a vector field point of view; see Greenfield [Gr] and Hill and Taiani [HT]. One must, however, use (1.4) to go from the point of view described here to the vector field point of view.

3. Levi geometry in higher codimension. Suppose S is a generic submanifold of \mathbb{C}^n . The real cotangent bundle of S , $T^*(S) = T^*(S) \otimes 1$, can be imbedded into its complexification $CT^*(S) = T^*(S) \otimes_{\mathbb{R}} \mathbb{C}$. We let $CH^*(S)$ and $\mathbb{C}\mathcal{G}$ denote the complexified holomorphic cotangent bundle and the complexified totally real subbundle of S , respectively. Using the complex hermitian innerproduct, $CH^*(S)$ orthogonally decomposes into $CH^*(S) = H^{1,0}(S) \oplus H^{0,1}(S)$ where $H^{1,0}(S)$ and $H^{0,1}(S)$ are, respectively, the $+i$ and $-i$ eigenspaces of the complex structure map $J: CH^*(S) \rightarrow CH^*(S)$. Thus, $CT^*(S)$ has the orthogonal splitting

$$(3.1) \quad CT^*(S) = H^{1,0}(S) \oplus H^{0,1}(S) \oplus \mathbb{C}\mathcal{G}.$$

We make the following definition of a CR-function on S , which is useful for our purposes.

3.2. DEFINITION. Suppose $V \subset S$ is an open set. A smooth function $f: V \rightarrow \mathbb{C}$ is called a CR-function on V if $(d_s f)(x)$ is orthogonal to $H_x^{0,1}(S)$ for each $x \in V$.

If S is locally defined by $S = \{x \in \mathbb{C}^n; r_1(x) = \dots = r_k(x) = 0\}$, where each $r_j: \mathbb{C}^n \rightarrow \mathbb{R}$ is smooth and $dr_1 \wedge \dots \wedge dr_k \neq 0$ on S , then the above definition is equivalent to $\bar{\partial} \tilde{f} \wedge \bar{\partial} r_1 \wedge \dots \wedge \bar{\partial} r_k = 0$ on V , where \tilde{f} is any smooth ambient extension of f . The fact we need about real analytic CR-functions is the following result due to Tomassini [T] (see also Greenfield [Gr, p. 310]).

3.3. THEOREM. *Suppose S is a real analytic generic submanifold. Suppose f is a real analytic CR-function on some nonempty open subset $V \subset S$. Then f is the restriction to V of a unique holomorphic function defined on some neighborhood of V in \mathbb{C}^n .*

The purpose of this section is to use Tomassini's theorem to prove the following geometric result:

3.4. THEOREM. *Suppose S is a real analytic, regular submanifold of \mathbb{C}^n of real dimension $2n - k$. Suppose $\text{ex dim}(S) = k - p$, $0 \leq p \leq k$, and suppose $x_0 \in S$. Then there exist p transverse, real analytic, Levi flat hypersurfaces N_1, \dots, N_p which are Levi flat near x_0 with $S \subset N_1 \cap \dots \cap N_p$. Moreover, there exist functions F_1, \dots, F_p which are holomorphic near x_0 , such that $N_i = \{x \in \mathbb{C}^n; \text{Re } F_i(x) = 0\}$, $1 \leq i \leq p$. The submanifold $N_1 \cap \dots \cap N_p$ is a real analytic Levi flat submanifold of codimension p which is unique in the following sense. If $\tilde{N}_1, \dots, \tilde{N}_p$ are transverse Levi flat, real analytic hypersurfaces with $S \subset \tilde{N}_1 \cap \dots \cap \tilde{N}_p$, then there is a neighborhood U of x_0 in \mathbb{C}^n with $N_1 \cap \dots \cap N_p \cap U = \tilde{N}_1 \cap \dots \cap \tilde{N}_p \cap U$.*

PROOF. Using the notation from §§1 and 2, we let $\delta_l: \mathcal{G}_{l-1} \rightarrow \Lambda^2 \Gamma^*(S) / \mathcal{G}_{l-1} \wedge \Gamma^*(S)$ be the l th Levi form on S and $\mathcal{G}_l = \ker \delta_l$. The assumptions on S imply that

$\mathcal{G}_{\text{int}} = \bigcap_{l=1}^{\infty} \mathcal{G}_l$ is the maximal integrable subbundle of \mathcal{G} of real dimension p . (Note that $k - p = \text{ex dim}(S) = \text{dim}(\mathcal{G}/\mathcal{G}_{\text{int}}) \Rightarrow \text{dim } \mathcal{G}_{\text{int}} = p$.)

The idea is to find $v_1, \dots, v_p: S \rightarrow \mathbf{R}$ which are real analytic and CR near x_0 , with $\text{sp}\{d_S v_1, \dots, d_S v_p\} = \mathcal{G}_{\text{int}}$. Then we shall use Tomassini's theorem to find F_1, \dots, F_p which are holomorphic near x_0 with $F_j|_S = iv_j$, $1 \leq j \leq p$. The hypersurfaces $N_j = \{x \in \mathbf{C}^n; \text{Re } F_j(x) = 0\}$, $1 \leq j \leq p$, will be the desired Levi flat hypersurfaces (note: $i = \sqrt{-1}$).

Now, \mathcal{G}_{int} is an integrable, real analytic subbundle of \mathcal{G} (i.e. $d_S \mathcal{G}_{\text{int}} \subset \mathcal{G}_{\text{int}} \wedge T^*(S)$). By the real Froebenius theorem, there exist $v_1, \dots, v_p: S \rightarrow \mathbf{R}$ which are real analytic near x_0 such that $\text{sp}\{d_S v_1, \dots, d_S v_p\} = \mathcal{G}_{\text{int}}$ near x_0 , and $v_1(x_0) = \dots = v_p(x_0) = 0$. In particular, $d_S v_1 \wedge \dots \wedge d_S v_p \neq 0$. From the point of view of vector fields, recall that $\mathcal{G}_{\text{int}} = \{X \in T(S); (\eta, X) = 0 \text{ for each } \eta \in \mathcal{G}_{\text{int}}\}$. The integrability of \mathcal{G}_{int} implies that \mathcal{G}_{int} is involutive, and the functions v_1, \dots, v_p found above satisfy $X(v_j) = 0$ for each $X \in \mathcal{G}_{\text{int}}$ and $1 \leq j \leq p$.

Since $d_S v_j \in \mathcal{G}_{\text{int}} \subset \mathcal{G} \subset \mathbf{C}\mathcal{G}$, each v_j is a CR-function on S near x_0 by Definition 3.2 and by the orthogonal decomposition given in (3.1). So, by Theorem 3.3, iv_j is the restriction of a function $F_j(F_j|_S = iv_j)$ which is holomorphic in some neighborhood in \mathbf{C}^n containing x_0 . Let $R_j = \text{Re } F_j$ and $I_j = \text{Im } F_j$, $1 \leq j \leq p$. The hypersurfaces $N_j = \{x \in \mathbf{C}^n; R_j(x) = 0\}$, $1 \leq j \leq p$, are clearly real analytic and Levi flat. Since each F_j is purely imaginary on S , clearly $S \subset N_1 \cap \dots \cap N_p$. To ensure that N_1, \dots, N_p are transverse, we only need to check that $dR_1 \wedge \dots \wedge dR_p \neq 0$ at x_0 . But if $dR_1 \wedge \dots \wedge dR_p = 0$ at x_0 , then certainly $dI_1 \wedge \dots \wedge dI_p = 0$ at x_0 , because the Cauchy-Riemann equations imply that $dI_j = -JdR_j$. Since $I_j|_S = v_j$, then $\{dI_1 \wedge \dots \wedge dI_p = 0 \text{ at } x_0\} \Rightarrow \{d_S v_1 \wedge \dots \wedge d_S v_p = 0 \text{ at } x_0\}$, which is a contradiction to the choice of v_1, \dots, v_p . Therefore, $dR_1 \wedge \dots \wedge dR_p \neq 0$ at x_0 as desired.

To prove uniqueness, suppose $\tilde{F}_j = (\tilde{R}_j + i\tilde{I}_j)$, $1 \leq j \leq p$, are holomorphic functions near x_0 , with $S \subset \{x \in \mathbf{C}^n; \tilde{R}_1(x) = \dots = \tilde{R}_p(x) = 0\}$ and $d\tilde{R}_1 \wedge \dots \wedge d\tilde{R}_p \neq 0$ on S . Let $\tilde{v}_j = \tilde{I}_j|_S$. By adding a purely imaginary number to each \tilde{F}_j , we may assume $\tilde{v}_j(x_0) = 0$. Clearly, $d_S \tilde{v}_1 \wedge \dots \wedge d_S \tilde{v}_p \neq 0$ on S near x_0 . We claim that $d_S \tilde{v}_j \in \mathcal{G}_{\text{int}}$ for each $1 \leq j \leq p$. To see this, note that $d\tilde{I}_j|_S = -Jd\tilde{R}_j|_S \in \mathcal{G} \subset T^*(S)$. Therefore $d_S\{\tilde{I}_j|_S\} = d\tilde{I}_j|_S$, and so $d_S \tilde{v}_j \in \mathcal{G}$. Since $d_S(d_S \tilde{v}_j) = 0$, it is clear that $\delta_l(d_S \tilde{v}_j) = 0$ for each $l \geq 1$ and each $1 \leq j \leq p$. Therefore $d_S \tilde{v}_j \in \bigcap_{l=1}^{\infty} \mathcal{G}_l = \mathcal{G}_{\text{int}}$, for each $1 \leq j \leq p$, which proves our claim.

To sum up, we have the real analytic functions v_1, \dots, v_p and $\tilde{v}_1, \dots, \tilde{v}_p$ with $\text{sp}\{d_S v_1, \dots, d_S v_p\} = \mathcal{G}_{\text{int}} = \text{sp}\{d_S \tilde{v}_1, \dots, d_S \tilde{v}_p\}$. This means that each \tilde{v}_j is a constant function on the level sets $V_t = \{x \in S; v_1(x) = t_1, \dots, v_p(x) = t_p\}$ for each $t = (t_1, \dots, t_p) \in \mathbf{R}^p$ near the origin. Conversely, each v_j is a constant function on the level sets $\tilde{V}_t = \{x \in S; \tilde{v}_1(x) = t_1, \dots, \tilde{v}_p(x) = t_p\}$. Therefore, there exist functions $f, g: \mathbf{R}^p \rightarrow \mathbf{R}^p$ which are real analytic near the origin such that $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_p) = f(v_1, \dots, v_p)$ and $v = (v_1, \dots, v_p) = g(\tilde{v}_1, \dots, \tilde{v}_p)$. If f_j is the j th component function of f , $1 \leq j \leq p$, then f_j has a power series expansion

$$\tilde{v}_j = f_j(v_1, \dots, v_p) = \sum_{\alpha = \alpha_1 \dots \alpha_p} a_{\alpha}^j v^{\alpha} \quad (a_{\alpha}^j \in \mathbf{R})$$

which is valid in some neighborhood of the origin. We claim that

$$\begin{aligned} \tilde{F}_j &= if_j(-iF_1, \dots, -iF_p) \\ &= i \sum_{\alpha=\alpha_1, \dots, \alpha_p} a_\alpha^j (-iF_1)^{\alpha_1} \cdots (-iF_p)^{\alpha_p} \end{aligned}$$

near x_0 . To see this, we note that both \tilde{F}_j and $if_j(-iF_1, \dots, -iF_p)$ are holomorphic functions near S which equal $i\tilde{v}_j$ on S . Therefore, $\tilde{F}_j = if_j(-F_1, \dots, -iF_p)$ by the uniqueness part of Theorem 3.3.

We now compute

$$\begin{aligned} \tilde{R}_j &= \operatorname{Re} \left\{ i \sum_{\alpha=\alpha_1, \dots, \alpha_p} a_\alpha^j (-i)^{|\alpha|} (R_1 + iI_1)^{\alpha_1} \cdots (R_p + iI_p)^{\alpha_p} \right\} \\ &= - \sum_{\alpha=\alpha_1, \dots, \alpha_p} a_\alpha^j \operatorname{Im} \left\{ (-i)^{|\alpha|} (R_1 + iI_1)^{\alpha_1} \cdots (R_p + iI_p)^{\alpha_p} \right\} \end{aligned}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_p$. First we note that the constant term of this power series vanishes because $\tilde{R}_j = 0$ on S . Next, we note that the terms of this power series which just involve I_1, \dots, I_p are

$$- \sum_{\alpha=\alpha_1, \dots, \alpha_p} a_\alpha^j \operatorname{Im} \left\{ (-i)^{|\alpha|} (i)^{|\alpha|} I_1^{\alpha_1} \cdots I_p^{\alpha_p} \right\},$$

which vanish. Therefore, all the terms in the power series for \tilde{R}_j involve at least one factor of one of the functions R_q , $1 \leq q \leq p$. In particular, there exists a neighborhood U of x_0 in \mathbf{C}^n (governed by the domain of convergence of f) such that

$$\{x \in U; R_1(x) = \cdots = R_p(x) = 0\} \subset \{x \in U; \tilde{R}_1(x) = \cdots = \tilde{R}_p(x) = 0\}.$$

The above argument can be repeated for g to get the reverse inclusion. This proves our uniqueness assertion. Q.E.D.

REMARK. An analogous result in the C^∞ category is also valid. Namely, we have

3.4'. THEOREM. *Suppose S satisfies the hypothesis of Theorem 3.4, except we only assume that S is smooth. Then, locally, there exist hypersurfaces $N_j = \{x \in \mathbf{C}^n; r_j(x) = 0\}$ where each $r_j: \mathbf{C}^n \rightarrow \mathbf{R}$ is smooth, $1 \leq j \leq p$, with $dr_1 \wedge \cdots \wedge dr_p \neq 0$ on S such that $S \subset N_1 \cap \cdots \cap N_p$; and furthermore, for each $1 \leq j \leq p$, $\partial\bar{\partial}r_j$ vanishes to infinite order at S .*

The proof for Theorem 3.4 is also valid for Theorem 3.4' except that instead of using Tomassini's result (Theorem 3.3), we use the following result due to Nirenberg and Wells [NW] and also [N]: A smooth CR-function on a smooth submanifold S is locally the restriction to S of a smooth function f such that $\bar{\partial}f$ vanishes to infinite order on S .

In §5, we will need the following lemma, which shows that a smooth generic submanifold which is strictly pseudoconvex at a point is locally contained in a strictly pseudoconvex (real) hypersurface.

3.5. LEMMA. *Suppose S is a smooth generic submanifold of \mathbf{C}^n of real dimension $2n - k$. Let $x_0 \in S$ and suppose S is strictly pseudoconvex at x_0 . Then there exists a hypersurface $M = \{x \in \mathbf{C}^n; \rho(x) = 0\}$, $d\rho \neq 0$, $\rho: \mathbf{C}^n \rightarrow \mathbf{R}$ with $S \subset M$ and such that ρ is strictly plurisubharmonic on a neighborhood of x_0 in \mathbf{C}^n .*

PROOF. As before, let \mathcal{G} be the totally real subbundle of $T^*(S)$. Since S is strictly pseudoconvex at x_0 , there exists a one form $\eta \in \mathcal{G}$ of unit length and a positive constant C such that

$$(3.6) \quad \langle d\eta, \omega \wedge J\omega \rangle_{x_0}^{\mathbf{R}} \geq C |\omega|^2$$

for $\omega \in H_{x_0}(S)$.

We can, locally, construct a smooth function $\rho_1: \mathbf{C}^n \rightarrow \mathbf{R}$, which vanishes on S and satisfies $d\rho_1 = -J\eta$ on S , near x_0 . The function ρ_1 can be constructed as follows:

For $p \in S$, let $N_p^*(S)$ be the orthogonal complement (real Euclidean metric) to $T_p^*(S)$. Clearly $\dim_{\mathbf{R}} N_p(S) = k$ and $(J\eta)(p) \in N_p^*(S)$. Let $v_2, \dots, v_k \in N^*(S)$ so that for each $p \in S$ near x_0 , $\{(J\eta)(p), v_2(p), \dots, v_k(p)\}$ form an orthonormal basis for $N_p^*(S)$. Let $\phi: \mathbf{R}^{2n-k} \rightarrow S$ be a local parameterization for S near x_0 . Consider $\Phi: \mathbf{R}^{2n-k} \times \mathbf{R}^{k-1} \rightarrow \mathbf{R}^{2n}$,

$$\Phi(x_1 \cdots x_{2n-k}, t_2 \cdots t_k) = \phi(x) + t_2 v_2(\phi(x)) + \cdots + t_k v_k(\phi(x)).$$

Clearly Φ locally parameterizes a real hypersurface M_1 which contains S (because $\Phi(x, 0) = \phi(x)$) and such that $(J\eta)(p)$ is orthogonal to $T_p^*(M_1) = T_p^*(S) \oplus \text{sp}\{v_2(p), \dots, v_k(p)\}$. Let ρ_1 be a local defining function for M_1 normalized so that $d\rho_1$ has unit length. Then clearly $d\rho_1|_S$ (or $-d\rho_1|_S$) equals $-J\eta$.

The inequality in (3.6) can be written as $\langle \partial\bar{\partial}\rho_1, W \wedge \bar{W} \rangle_{x_0}^{\mathbf{C}^n} \geq C |W|^2$ for $W \in H_{x_0}^{1,0}(S)$. This implies the complex Hessian of ρ_1 at x_0 has $n - k$ positive eigenvalues $\lambda_1, \dots, \lambda_{n-k}$. From here on, we follow the standard argument (see Hunt and Wells [HW]) to obtain a strictly plurisubharmonic defining function. First, we find smooth functions $\rho_2, \dots, \rho_k: \mathbf{C}^n \rightarrow \mathbf{R}$ such that, near x_0 , $S = \{x \in \mathbf{C}^n; \rho_1(x) = \cdots = \rho_k(x) = 0\}$ and $d\rho_1 \wedge \cdots \wedge d\rho_k \neq 0$ on S . Next, we make the biholomorphic change of variables given in Hunt and Wells [HW, p. 810]. That is, we can arrange new holomorphic coordinates for \mathbf{C}^n , $\tau = (z, w)$, $z \in \mathbf{C}^k$, $w \in \mathbf{C}^{n-k}$, such that

$$\rho_j(z, w) = \text{Re } z_j + h_j(z, w), \quad 1 \leq j \leq k,$$

where each $h_j(z, w)$ is a smooth function satisfying

$$h_j(z, w) = \mathcal{O}(|\tau|^2), \quad 1 \leq j \leq k,$$

and, in particular,

$$h_1(z, w) = \sum_{j=1}^{n-k} \lambda_j |w_j|^2 + \mathcal{O}(|\tau|^3).$$

We have used the notation $h = \mathcal{O}(|\tau|^q)$ to mean that there are constants $C, \eta > 0$ such that $|h(\tau)| \leq C |\tau|^q$ for $|\tau| \leq \eta$. Note that, in the new coordinates, $x_0 = 0$, $\eta = dy_1$, and $H_0^{1,0}(S) = \{0\} \times \mathbf{C}^{n-k} \simeq \mathbf{C}^{n-k}$ with coordinates (w_1, \dots, w_{n-k}) .

We now choose positive real constants $\gamma_1, \dots, \gamma_k$ and make the following biholomorphic change of variables:

$$\begin{aligned} z'_1 &= z_1 + (\gamma_1 z_1^2 + \dots + \gamma_k z_k^2), \\ z'_j &= z_j, \quad 2 \leq j \leq k, \\ w'_j &= w_j, \quad 1 \leq j \leq n - k. \end{aligned}$$

It is clear that

$$\operatorname{Re} z_1 = x'_1 + \sum_{j=1}^k \gamma_j \left((y'_j)^2 - (x'_j)^2 \right) + \mathcal{O}(|\tau'|^3)$$

where we have set $x'_j = \operatorname{Re} z'_j$, $y'_j = \operatorname{Im} z'_j$ for $1 \leq j \leq k$. Therefore, in the new coordinates, we have

$$(3.7) \quad \rho'_1(z', w') = x'_1 + \sum_{j=1}^k \gamma_j \left((y'_j)^2 - (x'_j)^2 \right) + \sum_{j=1}^{n-k} \lambda_j |w'_j|^2 + \mathcal{O}(|\tau'|^3)$$

and

$$\rho'_j(z', w') = x'_j + h'_j(z', w') \quad \text{for } 1 \leq j \leq k,$$

with

$$h'_j(z', w') = \mathcal{O}(|\tau'|^2).$$

Now, on S , $x'_j = -h'_j(z', w') = \mathcal{O}(|\tau'|^2)$ for $1 \leq j \leq k$. Let $\rho''_1(z', w')$ be defined by setting $(x'_j)^2 = (h'_j(z', w'))^2$, $1 \leq j \leq k$, in the first sum appearing in (3.7). The function ρ''_1 also vanishes on S . Since $(h'_j(\tau'))^2 = \mathcal{O}(|\tau'|^4)$, we have

$$\begin{aligned} \rho''_1(z', w') &= x'_1 + \sum_{j=1}^k \gamma_j (y'_j)^2 + \sum_{j=1}^{n-k} \lambda_j |w'_j|^2 + \mathcal{O}(|\tau'|^3) \\ &= x'_1 - \frac{1}{2} \operatorname{Re} \left\{ \sum_{j=1}^k \gamma_j (z'_j)^2 \right\} + \frac{1}{2} \left[\sum_{j=1}^k \gamma_j |z'_j|^2 \right] + \sum_{j=1}^{n-k} \lambda_j |w'_j|^2 + \mathcal{O}(|\tau'|^3). \end{aligned}$$

Clearly, the complex hessian of ρ''_1 at the origin is positive definite with eigenvalues $\frac{1}{2}\gamma_1, \dots, \frac{1}{2}\gamma_k, \lambda_1, \dots, \lambda_{n-k}$. Strict plurisubharmonicity is an open condition; hence ρ''_1 is strictly plurisubharmonic in a neighborhood of the origin in \mathbb{C}^n . \square

Theorem 3.4 and Lemma 3.5 together imply the following result which we will need in §5.

3.8. THEOREM. *Suppose S is a real analytic, regular submanifold of (real) dimension $2n - k$. Suppose that $\dim(S) = k - p$, $0 \leq p \leq k - 1$, and that S is strictly pseudoconvex at a point $x_0 \in S$. Then there exist real analytic functions $r_1, \dots, r_p, \rho: \mathbb{C}^n \rightarrow \mathbb{R}$ and an open set U in \mathbb{C}^n containing x_0 with the following properties: (a) if $M = \{x \in \mathbb{C}^n; \rho(x) = 0\}$ and $N_j = \{x \in \mathbb{C}^n; r_j(x) = 0\}$, $1 \leq j \leq p$, then $S \cap U \subset M \cap N_1 \cap \dots \cap N_p$ and $d\rho \wedge dr_1 \wedge \dots \wedge dr_p \neq 0$ on $S \cap U$; (b) ρ is strictly plurisubharmonic on U ; (c) each r_j is the real part of a holomorphic function on U .*

PROOF. Using Theorem 3.4, we may first find transverse hypersurfaces $N_j = \{x \in \mathbb{C}^n; r_j(x) = 0\}$, $1 \leq j \leq p$, such that $S \subset N_1 \cap \dots \cap N_p$ and each r_j is locally the real part of a holomorphic function. Then using Lemma 3.5, we may find a hypersurface

$M = \{x \in \mathbf{C}^n; \rho(x) = 0\}$ such that $S \subset M$ and ρ is strictly plurisubharmonic near x_0 in \mathbf{C}^n . Of course, the biholomorphisms occurring in the proof of Lemma 3.5 preserve the pluriharmonicity of each r_j . Thus, we need only show that $d\rho \wedge dr_1 \wedge \dots \wedge dr_p \neq 0$ near x_0 . As before, let $\mathcal{G}_{\text{int}} = \bigcap_{l=1}^{\infty} \mathcal{G}_l$, where

$$\mathcal{G}_l = \ker \delta_l: \mathcal{G}_{l-1} \rightarrow \Lambda^2 \Gamma^*(S) / \mathcal{G}_{l-1} \wedge T^*(S).$$

Recall from the proof of Theorem 3.4 that $\dim_{\mathbf{R}} \mathcal{G}_{\text{int}} = p$ and $\mathcal{G}_{\text{int}} = \text{sp}\{Jdr_1|_S, \dots, Jdr_p|_S\}$. If $d\rho \wedge dr_1 \wedge \dots \wedge dr_p = 0$ at x_0 , then

$$Jd\rho(x_0) \in \text{sp}\{Jdr_1(x_0), \dots, Jdr_p(x_0)\}.$$

In particular, $\delta_1(Jd\rho)(x_0) = 0$. But since ρ is strictly plurisubharmonic, clearly $\delta_1(Jd\rho)(x_0) \neq 0$. Thus $d\rho \wedge dr_1 \wedge \dots \wedge dr_p \neq 0$ at x_0 , as desired. Q.E.D.

4. Kernels and formal identities. In this section we describe the kernels we shall use and some formal identities which relate these kernels to the tangential Cauchy-Riemann equations. Much of this material is presented thoroughly in Harvey and Polking [HP], so we shall be brief. First, we need some more notation.

Let $\Lambda^{p,q}(\mathbf{C}^n)$ denote the bundle of forms on \mathbf{C}^n of bidegree p, q . Smooth sections of $\Lambda^{p,q}(\mathbf{C}^n)$ over an open set $U \subset \mathbf{C}^n$ will be denoted by $\Gamma^{p,q}(U)$.

Suppose S is a smooth generic submanifold of \mathbf{C}^n of real dimension $2n - k$, locally defined by $S = \{x \in \mathbf{C}^n; \rho_1(x) = \dots = \rho_k(x) = 0\}$ with $d\rho_1 \wedge \dots \wedge d\rho_k \neq 0$ near S . We let I_ρ be the ideal in $\Lambda^*(\mathbf{C}^n)$ generated by $\bar{\partial}\rho_1, \dots, \bar{\partial}\rho_k$. Since S is generic, I_ρ is a well-defined ideal of constant dimension defined over some set in \mathbf{C}^n containing S . The restriction of this ideal to S , $I_\rho|_S$, is called the ideal of complex normal forms on S . Using the complex hermitian metric, we let I_ρ^\perp be the orthogonal complement of I_ρ in $\Lambda^*(\mathbf{C}^n)$.

On S we define $\Lambda^{p,q}(\mathbf{C}^n)|_S$ to be the restriction of the bundle $\Lambda^{p,q}(\mathbf{C}^n)$ to S . An element in $\Lambda^{p,q}(\mathbf{C}^n)|_S$ is an ambient p, q form whose coefficients have been restricted to S . We also define

$$\Lambda_S^{p,q} = \{\Lambda^{p,q}(\mathbf{C}^n)|_S \cap I_\rho^\perp|_S\},$$

and we call this the bundle of complex tangential (p, q) forms over S . For V an open set in S , we let $\Gamma_S^{p,q}(V)$ be the space of smooth sections of $\Lambda_S^{p,q}$ over V and we let $\mathcal{D}_S^{p,q}(V)$ be the space of sections in $\Gamma_S^{p,q}(V)$ with compact support in V .

There is a well-defined orthogonal projection map $T_S: \Lambda^*(\mathbf{C}^n) \rightarrow I_\rho^\perp$, and we shall often write f_{T_S} for $T_S(f)$. If $f \in \Lambda_S^{p,q}(\mathbf{C}^n)|_S$, then $f_{T_S} \in \Lambda_S^{p,q}$ and we call f_{T_S} the complex tangential piece of f along S . It must be kept in mind that if $f \in \Lambda^*(\mathbf{C}^n)$, then f_{T_S} is defined on an open neighborhood of S in \mathbf{C}^n .

The induced Cauchy-Riemann operator $\bar{\partial}_S: \Gamma_S^{p,q} \rightarrow \Gamma_S^{p,q+1}$ can now be defined as follows: Suppose $f \in \Gamma_S^{p,q}(V)$; choose $\tilde{f} \in \Gamma^{p,q}(\mathbf{C}^n)$ with $\tilde{f}_{T_S} = f$ on V ; then $\bar{\partial}_S f = (\bar{\partial}\tilde{f}|_S)_{T_S} \in \Gamma_S^{p,q+1}(V)$. It can be easily verified that this definition is independent of the choice of ambient extension.

If S is an orientable submanifold of \mathbf{C}^n of real codimension k , then the current integration over S' is a current of degree k defined by

$$([S], \phi)_{\mathbf{C}^n} = \int_S \phi$$

for $\phi \in \mathcal{O}^*(\mathbf{C}^n)$. This current orthogonally splits into its bidegrees, i.e.

$$[S] = \sum_{p+q=k} [S]^{p,q}.$$

The most important piece of this splitting is $[S]^{0,k}$.

We have

$$([S]^{0,k}, \phi)_{\mathbf{C}^n} = \int_S \phi$$

where it is understood that only the piece of ϕ of bidegree $(n, n - k)$ is integrated. The important relation between $[S]^{0,k}$ and $\bar{\partial}_S$ is the following: if $f \in \Gamma_S^{p,q}$, then $\bar{\partial}_S f = g$ on S if and only if $\bar{\partial} \tilde{f} \wedge [S]^{0,k} = \tilde{g} \wedge [S]^{0,k}$ where \tilde{f} and \tilde{g} are ambient extensions of f and g , respectively (see [AB]).

Let U be an open set in \mathbf{C}^n . Suppose $K(\zeta, z)$ is a form on $\mathbf{C}^n \times \mathbf{C}^n$ with smooth coefficients $S \times U$. Suppose $f \in \mathcal{O}_S^{p,q}(S)$. Then we let

$$K([S]^{0,k} \wedge f)(z) = \int_{\zeta \in S} K(\zeta, z) \wedge \tilde{f}(\zeta),$$

where \tilde{f} is any ambient extension of f , and it is understood that only the piece of $K(\zeta, z) \wedge \tilde{f}(\zeta)$ of bidegree $(n, n - k)$ is integrated. It is clear that $K([S]^{0,k} \wedge f)(z)$ is a smooth form on U . In fact, if the bidegree of K (in both ζ and z) is $(n, n - r)$, then $K: \mathcal{O}_S^{p,q} \rightarrow \Gamma^{p,q-r+k}(U)$. We also will need the following fundamental identity (see [HP, Theorem 2.8]):

$$(4.1) \quad (\bar{\partial}K)([S]^{0,k} \wedge f) = \bar{\partial}\{K([S]^{0,k} \wedge f)\} + (-1)^{r+k}K([S]^{0,k} \wedge \bar{\partial}_S f),$$

where the $\bar{\partial}$ on the left-hand side is in both the ζ and z variables.

Now, suppose M is a generic submanifold of \mathbf{C}^n (later we will take M to be a hypersurface). Let V be an open subset of $M \times M$ with coordinates (ζ, z) . Suppose $u^j(\zeta, z) = (u_1^j(\zeta, z), \dots, u_n^j(\zeta, z))$: $V \rightarrow \mathbf{C}^n$ is a smooth mapping for $1 \leq j \leq N$. We adopt the following notation from [HP]:

$$\begin{aligned} u^j(\zeta, z) \cdot (\zeta - z) &= \sum_{k=1}^n u_k^j(\zeta, z)(\zeta_k - z_k), \\ u^j(\zeta, z) \cdot d(\zeta - z) &= \sum_{k=1}^n u_k^j(\zeta, z)d(z_k - z_k), \\ \bar{\partial}_M u^j(\zeta, z) \cdot d(\zeta - z) &= \sum_{k=1}^n \bar{\partial}_M u_k^j(\zeta, z) \wedge d(\zeta_k - z_k), \end{aligned}$$

where $\bar{\partial}_M$ is taken in both variables ζ and z . We will use the following one forms:

$$\alpha_j = \left[\frac{u^j(\zeta, z) \cdot d(\zeta - z)}{u^j(\zeta, z) \cdot (\zeta - z)} \right]_{T_{M \times M}}, \quad 1 \leq j \leq N.$$

Each α_j is smooth on $V - A_j$ where

$$A_j = \{(\zeta, z) \in V; u^j(\zeta, z) \cdot (\zeta - z) = 0\}.$$

4.2. PROPOSITION. For each integer $k \geq 1$, we have

$$\alpha_j \wedge (\bar{\partial}_M \alpha_j)^k = \left[\frac{u^j \cdot d(\zeta - z)}{u^j \cdot (\zeta - z)} \right] \wedge \left[\frac{\bar{\partial}_M u^j \cdot d(\zeta - z)}{u^j \cdot (\zeta - z)} \right]^k$$

on $V - A_j$.

PROOF. The proof is clear after noting that

$$\begin{aligned} \bar{\partial}_M \alpha_j &= \left[\frac{\bar{\partial}_M u^j \cdot d(\zeta - z)}{u^j(\zeta, z) \cdot (\zeta - z)} \right]_{T_{M \times M}} \\ &\quad - \left[\frac{\bar{\partial}_M \{u^j \cdot (\zeta - z)\} \wedge u^j \cdot d(\zeta - z)}{(u^j(\zeta, z) \cdot (\zeta - z))^2} \right]_{T_{M \times M}}. \quad \text{Q.E.D.} \end{aligned}$$

We shall also use the following multi-index notation:

If $\beta = (\beta_1, \dots, \beta_N)$ is an N -tuple of nonnegative integers, then we let

$$|\beta| = \beta_1 + \dots + \beta_N; \quad (\bar{\partial}_M \alpha)^\beta = (\bar{\partial}_M \alpha_1)^{\beta_1} \wedge \dots \wedge (\bar{\partial}_M \alpha_N)^{\beta_N}.$$

If $I = \{i_1, \dots, i_p\}$ is a p -tuple of positive integers, then we let $\alpha_I = \alpha_{i_1} \wedge \dots \wedge \alpha_{i_p}$ and

$$E_I(u^{i_1}, \dots, u^{i_p}) = (2\pi i)^{-n} \left\{ \alpha_I \wedge \sum_{\substack{|\beta| = n-p \\ \beta = (\beta_1, \dots, \beta_p)}} (\bar{\partial}_M \alpha)^\beta \right\}.$$

If the u^i are understood, then we just write E_I for $E_I(u^{i_1}, \dots, u^{i_p})$. The form E_I is a smooth form on $V - \{\cup_{j=1}^p A_{i_j}\}$.

For $1 \leq j \leq p$, we set $I_j = \{i_1, \dots, \widehat{i_j}, \dots, i_p\}$ where we mean that the element i_j is omitted. The formal identity we need from Harvey and Polking [HP, Theorem 4.10] is the following:

4.3. THEOREM. On the set $V - \{\cup_{j=1}^p A_{i_j}\}$, we have

$$\bar{\partial}_M E_I = \sum_{j=1}^p (-1)^j E_{I_j}$$

where, as before, $\bar{\partial}_M$ is taken in both ζ and z .

In the next section, we will take $p = 4$ and construct u^1, u^2, u^3, u^4 so that the resulting kernels defined above will locally solve the tangential Cauchy-Riemann equations.

5. The local solution to $\bar{\partial}_S$ in codimension two. In this section, we gather together the notation and identities from §4 and the geometric results from §3 and prove our local solvability result.

5.1. THEOREM. Suppose S is a real analytic regular submanifold of \mathbb{C}^n of real dimension $2n - 2$. Suppose $\text{ex dim}(S) = 1$ and that S is strictly pseudoconvex at a point $x_0 \in S$. Then there exists a local neighborhood basis of open sets for S , $\{\omega_\alpha\}$, about x_0 such that the following holds: For each ω_α , there is an operator $K: \Gamma_S^{p,q}(\bar{\omega}_\alpha) \rightarrow \Gamma_S^{p,q-1}(\omega_\alpha)$ such that if $f \in \Gamma_S^{p,q}(\bar{\omega}_\alpha)$, $1 \leq q \leq n - 3$, and $\bar{\partial}_S f = 0$ on ω_α , then $\bar{\partial}_S\{K(f)\} = f$ on ω_α .

This theorem provides a Poincaré lemma for $\bar{\partial}_S$ and, as will be seen from the proof, an explicit way of calculating the solution with kernels. The assumption that $f \in \Gamma_S^{p,q}(\bar{\omega}_\alpha)$ means that f is smooth on $\bar{\omega}_\alpha$.

Most of the rest of this section will be devoted to the proof of this theorem.

Using Theorem 3.8, we may find hypersurfaces $M = \{z \in \mathbb{C}^n; \rho(z) = 0\}$ and $N = \{z \in \mathbb{C}^n; r(z) = 0\}$ with $S = M \cap N$. We may assume ρ is strictly plurisubharmonic near x_0 and $\partial\bar{\partial}r = 0$ near x_0 in \mathbb{C}^n . By a local biholomorphic change of variables (see Hörmander [Ho, proof of Theorem 2.6.13]) we may assume that $x_0 = 0$ and that $M \cap U$ is a strictly convex hypersurface in U , where U is some open ball on \mathbb{C}^n containing $x_0 = 0$. In fact, we may choose local holomorphic coordinates so that

$$\rho(z) = \text{Re } z_1 - v(\text{Im } z_1, z_2, \dots, z_n)$$

where v is a strictly convex function near the origin with $v(0) = (\nabla v)(0) = 0$. (v is strictly convex near the origin if and only if the real Hessian of v at the origin is positive definite.) By possibly shrinking U , we may find a holomorphic function h on U with $\text{Re } h = r$ on U .

Let $W_\alpha = U \cap M \cap \{z \in \mathbb{C}^n; \text{Re } z_1 < \alpha\}$. Since v is a strictly convex function, it is clear that the collection of open sets $\{W_\alpha, \alpha > 0\}$ forms a local neighborhood basis for M about the point $x_0 = 0$. Therefore, the collection $\{\omega_\alpha = W_\alpha \cap S, \alpha > 0\}$ forms a local neighborhood basis of open sets for S about the point $x_0 = 0$.

From now on we fix $\alpha > 0$ small enough so that $W_\alpha \subset \subset U$.

We now wish to construct functions $u^1, u^2, u^3, u^4: U \times U \rightarrow \mathbb{C}^n$ so that the resulting kernels constructed in §4 can be used in the proof of Theorem 5.1. We let

$$\begin{aligned} u^1(\zeta, z) &= (\partial\rho(\zeta)/\partial\zeta_1, \dots, \partial\rho(\zeta)/\partial\zeta_n), \\ u^2(\zeta, z) &= (\partial\rho(z)/\partial z_1, \dots, \partial\rho(z)/\partial z_n). \end{aligned}$$

So u^1 is just the complex gradient of ρ at ζ , and we will think of u^1 and u^2 as depending on both variables (ζ, z) .

An easy power series argument shows that there exists a holomorphic mapping $v: U \times U \rightarrow \mathbb{C}^n$, $v = (v_1, \dots, v_n)$, such that

$$h(\zeta) - h(z) = \sum_{j=1}^n v_j(\zeta, z)(\zeta_j - z_j) = v(\zeta, z) \cdot (\zeta - z)$$

for $(\zeta, z) \in U \times U$. We shall let $u^3 = v$. Finally, we shall let $u^4(\zeta, z)$ be the constant vector $(1, 0, \dots, 0)$.

We shall need the following kernels constructed in §4: $E_{123}, E_{13}, E_{23}, E_{1234}, E_{134}, E_{234}, E_{124}$. Using Proposition 4.2, and the fact that $\bar{\partial}u^3 = \bar{\partial}u^4 = 0$, we can write out

these kernels. For example:

$$E_{134}(\zeta, z) = (2\pi i)^{-n} \left\{ \left[\frac{\partial \rho(\zeta)/\partial \zeta \cdot d(\zeta - z)}{\partial \rho(\zeta)/\partial \zeta \cdot (\zeta - z)} \right] \wedge \left[\frac{u^3(\zeta, z) \cdot d(\zeta - z)}{u^3(\zeta, z) \cdot (\zeta - z)} \right] \right. \\ \left. \wedge \frac{d(\zeta_1 - z_1)}{(\zeta_1 - z_1)} \wedge \left[\frac{\bar{\partial}_\zeta u^1 \cdot d(\zeta - z)}{u^1(\zeta, z) \cdot (\zeta - z)} \right]^{n-3} \right\}_{T_{M \times M}}, \tag{5.2}$$

$$E_{1234}(\zeta, z) = (2\pi i)^{-n} \left\{ \left(\frac{u^1 \cdot d(\zeta - z)}{u^1 \cdot (\zeta - z)} \right) \wedge \left(\frac{u^2 \cdot d(\zeta - z)}{u^2 \cdot (\zeta - z)} \right) \right. \\ \wedge \left(\frac{u^3 \cdot d(\zeta - z)}{u^3 \cdot (\zeta - z)} \right) \wedge \left(\frac{d(\zeta_1 - z_1)}{(\zeta_1 - z_1)} \right) \\ \left. \wedge \sum_{j+k=n-4} \left(\frac{\bar{\partial}_\zeta u^1 \cdot d(\zeta - z)}{u^1 \cdot (\zeta - z)} \right)^j \wedge \left(\frac{\bar{\partial}_z u^2 \cdot d(\zeta - z)}{u^2 \cdot (\zeta - z)} \right)^k \right\}_{T_{M \times M}}. \tag{5.3}$$

Let $V = \{M \times M\} \cap \{U \times U\}$. We have the following equation from Theorem 4.3:

$$\bar{\partial}_M E_{1234} = E_{123} - E_{124} + E_{134} - E_{234} \tag{5.4}$$

which holds on $V - \{\cup_{i=1}^4 A_i\}$, where

$$A_i = \{(\zeta, z) \in V; u^i(\zeta, z) \cdot (\zeta - z) = 0\}.$$

Let us carefully analyze A_1, A_2, A_3, A_4 . Since $M \cap U$ is strictly convex, u^1 is a strong support function for $M \cap U$, as defined in [AB, Definition 3.4]. In particular, this means that $u^1(\zeta, z)$ is holomorphic in z (u^1 is, in fact, independent of z) and that

$$A_1 = \{(\zeta, z) \in V; \partial \rho(\zeta)/\partial \zeta \cdot (\zeta - z) = 0\} = \{(z, z) \in V\}.$$

That is, A_1 is just the diagonal of V . Indeed, if $\zeta \in M$ and $\partial \rho(\zeta)/\partial \zeta \cdot (\zeta - z) = 0$, then $(\zeta - z)$ must be a holomorphic tangent vector of M at ζ ; and the strict convexity of $M \cap U$ implies that $z \notin M$; unless $z = \zeta$. Similarly,

$$A_2 = \{(\zeta, z) \in V; \partial \rho(z)/\partial z \cdot (\zeta - z) = 0\} = \{(z, z) \in V\} = A_1.$$

Now, $A_3 = \{(\zeta, z) \in V; h(\zeta) = h(z)\}$ and $A_4 = \{(\zeta, z) \in V; \zeta_1 = z_1\}$. Clearly, both A_1 and A_2 are contained in $A_3 \cap A_4$.

Let E denote any one of the kernels E_{123}, E_{13}, E_{23} . If $z \in U \cap M$ is fixed, then the ζ -singular support of $E(\zeta, z)$ on $M \cap U$ is contained in the set $A_3^z = \{\zeta \in M \cap U; h(\zeta) = h(z)\}$. For $|t|$ small, let $S_t = \{\zeta \in M \cap U; r(\zeta) = t\}$. If $r(z) = t$, then it is clear that $A_3^z \subset S_t$ because $r = \text{Re } h$. Therefore, if $f \in \mathcal{D}_S^{p,q}(S \cap U)$, then

$$E([S]^{0,2} \wedge f)(z) = \int_{\zeta \in S} E(\zeta, z) \wedge f(\zeta)$$

defines a smooth form for $z \in \{M^+ \cap U\} \cup \{M^- \cap U\}$, where $M^+ = \{z \in M; r(z) > 0\}$ and $M^- = \{z \in M; r(z) < 0\}$. We shall need the following boundary value result (see Theorem 5.3 in [AB]).

5.5. LEMMA. *Let E be any one of the kernels E_{123}, E_{13}, E_{23} . Suppose $f \in \mathcal{D}_S^*(S \cap U)$. Suppose N is any nonnegative integer and let $X_1 \cdots X_N$ be vector fields on M with $X_i(r) = 0$ for $1 \leq i \leq N$ (so $X_i|_S$ is a tangential vector field to S). Then $X_1 \cdots X_N\{E([S]^{0,2} \wedge f)\}|_{M^\pm \cap U}$ has a continuous extension to $S \cap U$.*

The content of Lemma 5.5 is that the boundary values, from either M^+ or M^- , of any tangential derivative of $E([S]^{0,2} \wedge f)$ exist on $S \cap U$. Certain stronger boundary value results are true; see [AB] for more details. We shall let $E^\pm(f)$ be the extension of $\{E([S]^{0,2} \wedge f)\}_{T_S}|_{M^\pm \cap U}$ to $S \cap U$. Lemma 5.5 implies that $E^\pm(f) \in \Gamma_S^*(S \cap U)$, i.e. $E^\pm(f)$ is a smooth tangential form on S .

Now we analyse the boundary behavior of $E([S]^{0,2} \wedge g)$ where $g \in \mathcal{D}_S^*(S \cap U)$ and E is any one of the kernels $E_{1234}, E_{124}, E_{134}, E_{234}$. A typical coefficient of $E(\zeta, z) \wedge g(\zeta)$ is

$$g(\zeta) / (u^1 \cdot (\zeta - z))^p (u^2 \cdot (\zeta - z))^q (u^3 \cdot (\zeta - z))^l (\zeta_1 - z_1)^k$$

where p, q, l, k are nonnegative integers. In general, $E([S]^{0,2} \wedge g)(z)$ is not even defined for $z \in \{M^+ \cap U\} \cap \{M^- \cap U\}$ because of the term $(\zeta_1 - z_1)^{-k}$. However, if $g \equiv 0$ on ω_α then the term

$$(5.6) \quad g(\zeta) / (u^1 \cdot (\zeta - z))^p (u^2 \cdot (\zeta - z))^q (\zeta_1 - z_1)^k$$

is smooth for $\zeta \in S$ and $z \in W_\alpha$. This is because the numerator is nonzero only when $\text{Re } \zeta_1 > \alpha$, and the denominators are nonvanishing when $\text{Re } z_1 < \alpha, \text{Re } \zeta_1 > \alpha$. Therefore, if $g \equiv 0$ on ω_α , then $E([S]^{0,2} \wedge g)$ is smooth on $W_\alpha - S$. Moreover, since $g(\zeta) / (\zeta_1 - z_1)^k$ is smooth for $\zeta \in S$ and $z \in W_\alpha$, the boundary behavior of $E([S]^{0,2} \wedge g)$ on $S \cap W_\alpha = \omega_\alpha$ is the same (in fact better) than the boundary behavior of $E_{123}([S]^{0,2} \wedge g)$. That is, the same proof in [AB] for Lemma 5.5 is valid for the following lemma.

5.5'. LEMMA. *Let E be any one of the kernels $E_{1234}, E_{134}, E_{124}, E_{234}$. Suppose $g \in \mathcal{D}_S^*(S \cap U)$ with $g \equiv 0$ on ω_α . Suppose N is any nonnegative integer and let $X_1 \cdots X_N$ be vector fields on M with $X_i(r) = 0$ for $1 \leq i \leq N$. Then $X_1 \cdots X_N\{E([S]^{0,2} \wedge g)\}|_{M^\pm \cap W_\alpha}$ has a continuous extension to $S \cap W_\alpha = \omega_\alpha$.*

Again, we shall let $E^\pm(g) \in \Gamma_S^*(\omega_\alpha)$ be the extension of $\{E([S]^{0,2} \wedge g)\}_{T_S}|_{M^\pm \cap W_\alpha}$ to ω_α .

As the next lemma shows, many of the kernels defined above act nontrivially only on forms having a particular bidegree.

5.7. LEMMA. (a) *Suppose $f \in \mathcal{D}_S^{p,q}(S \cap U)$; then:*

$$\begin{aligned} E_{13}([S]^{0,2} \wedge f) &= 0 \quad \text{unless } q = 0, \\ E_{23}([S]^{0,2} \wedge f) &= 0 \quad \text{unless } q = n - 2. \end{aligned}$$

(b) Suppose $g \in \mathcal{D}_S^{p,q}(S \cap U)$ and $g \equiv 0$ on ω_α ; then:

$$E_{134}([S]^{0,2} \wedge g) = 0 \quad \text{unless } q = 1,$$

$$E_{234}([S]^{0,2} \wedge g) = 0 \quad \text{unless } q = n - 2.$$

PROOF. For example, the kernel $E_{134}(\zeta, z)$ is written out in (5.2). Since u^1 does not depend on z , $E_{134}(\zeta, z)$ is of degree $n - 3$ in $d\bar{\zeta}$. Therefore, on S , $E_{134}(\zeta, z)$ will only pair with the piece of $g(\zeta)$ of type $(p, 1)$ (to add up to $n - 2$ $d\bar{\zeta}$'s). Similarly, the degree of E_{13} , E_{23} and E_{234} in $d\bar{\zeta}$ are, respectively, $n - 2$, zero, and zero. Q.E.D.

We must put the kernel E_{124} in a special category. Formally, E_{124} acts nontrivially on all bidegrees. However, soon we will be interested in the boundary value jump of these kernels across S . If $g \equiv 0$ on ω_α , then we claim that the boundary value jump of $E_{124}([S]^{0,2} \wedge g)$ across $S \cap W_\alpha = \omega_\alpha$ vanishes, i.e.,

$$(5.8) \quad (E_{124}^+ - E_{124}^-)(g) = 0 \quad \text{on } \omega_\alpha.$$

To see this, note that a typical term of $E_{124}(\zeta, z) \wedge g(\zeta)$, given in (5.6), is smooth for $\zeta \in S$ and $z \in W_\alpha$. This is because the term $u^3(\zeta, z) \cdot (\zeta - z)$ is missing from the denominator. Therefore, if $g \equiv 0$ on ω_α , then $E_{124}([S]^{0,2} \wedge g)$ is smooth on W_α (whereas, if E is any of the other kernels, then $E([S]^{0,2} \wedge g)$ is only a priori smooth on $W_\alpha - S$). Of course, the boundary value jump across $S \cap W_\alpha$ of a smooth form on W_α is zero.

We now summarize the "global" solution to $\bar{\partial}_S$ given in [AB]. let

$$E = E_{123}^+ - E_{123}^- : \mathcal{D}_S^{p,q}(S \cap U) \rightarrow \Gamma_S^{p,q-1}(S \cap U),$$

$$Q = (E_{13}^+ - E_{13}^-) + (E_{23}^- - E_{23}^+) : \mathcal{D}_S^{p,q}(S \cap U) \rightarrow \Gamma_S^{p,q}(S \cup U).$$

From formula (5.9) in §5 of [AB], we have

$$(5.9) \quad I = Q + \bar{\partial}_S \circ E + E \circ \bar{\partial}_S \quad (I = \text{identity}).$$

This holds as an operator equation from $\mathcal{D}_S^{p,q}(S \cap U)$ to $\Gamma_S^{p,q}(S \cap U)$. Actually, the operators E and Q constructed in [AB] act on the space $\mathcal{D}_S^{p,q}(S)$. But since h is only defined on U , the kernels E_{123} , E_{13} and E_{23} are only defined on $U \times U$. Hence these kernels only act on $\mathcal{D}_S^{p,q}(S \cap U)$.

If $f \in \mathcal{D}_S^{p,q}(S \cap U)$ with $1 \leq q \leq n - 3$, then $Q(f) = 0$ by Lemma 5.7, and we have

$$(5.10) \quad f = \bar{\partial}_S\{E(f)\} + E(\bar{\partial}_S f) \quad \text{on } S \cap U.$$

Note that this equation does not solve our local problem in Theorem 5.1, because we only assume $\bar{\partial}_S f = 0$ on ω_α (and not on all of $S \cap U$). However, we are now in a position to complete the proof of Theorem 5.1. Suppose first that $f \in \Gamma_S^{p,q}(\bar{\omega}_\alpha)$, $1 \leq q \leq n - 4$, and satisfies $\bar{\partial}_S f = 0$ on ω_α ; let $\tilde{f} \in \mathcal{D}_S^{p,q}(S \cap U)$ be a smooth extension of f . Clearly, $\text{supp}\{\bar{\partial}_M \tilde{f}\} \subset U \cap \{S - \omega_\alpha\}$. Using (5.10), we have

$$(5.11) \quad f = \tilde{f} = \bar{\partial}_S\{E(\tilde{f})\} + E(\bar{\partial}_S \tilde{f}) \quad \text{on } \omega_\alpha.$$

Using (5.4) we have

$$(5.12) \quad E_{123}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) = \bar{\partial}_M \{ E_{1234}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) \} \\ + (E_{124} - E_{134} + E_{234})([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}).$$

This equation makes sense on $W_\alpha - S$ since $\bar{\partial}_S \tilde{f} = 0$ on ω_α . We have also used equation (4.1) and the fact that $\bar{\partial}_S \bar{\partial}_S \tilde{f} = 0$. By Lemma 5.7,

$$E_{134}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) = E_{234}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) = 0.$$

So,

$$(5.13) \quad E_{123}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) = \bar{\partial}_M \{ E_{1234}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) \} + E_{124}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}).$$

As before, we let $S_t = \{ \zeta \in M \cap U; r(\zeta) = t \}$. If we apply the projection map T_S to equation (5.13) and then restrict this equation to $S_t, t \neq 0$, we obtain

$$\{ E_{123}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) \}_{T_S} = \bar{\partial}_{S_t} \{ E_{1234}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) \} + \{ E_{124}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) \}_{T_S}$$

on $S_t \cap W_\alpha$. We have used the fact that if F is a smooth form, then $\{ \bar{\partial}_M F \}_{T_S} = \bar{\partial}_{S_t} \{ F_{T_S} \}$ on S_t (see §4). Now $\bar{\partial}_{S_t}$ involves vector fields X which annihilate r ($X(r) = 0$). Therefore, by Lemmas 5.5 and 5.5', we can take boundary values of the above equation from $M^+ \cap W_\alpha$ or $M^- \cap W_\alpha$ (i.e., let $t \rightarrow 0^+$ or $t \rightarrow 0^-$). We obtain

$$E_{123}^\pm(\bar{\partial}_S \tilde{f}) = \bar{\partial}_S \{ E_{1234}^\pm(\bar{\partial}_S \tilde{f}) \} + E_{124}^\pm(\bar{\partial}_S \tilde{f}) \quad \text{on } \omega_\alpha.$$

Set $\mathcal{E} = E_{1234}^+ - E_{1234}^-$. By Lemma 5.5', $\mathcal{E}(\bar{\partial}_S \tilde{f}) \in \Gamma_S^{p, q-1}(\omega_\alpha)$. If we subtract $E_{123}^-(\bar{\partial}_S \tilde{f})$ from $E_{123}^+(\bar{\partial}_S \tilde{f})$ on ω_α and use (5.8), we obtain

$$(5.14) \quad E(\bar{\partial}_S \tilde{f}) = (E_{123}^+ - E_{123}^-)(\bar{\partial}_S \tilde{f}) = \bar{\partial}_S \{ \mathcal{E}(\bar{\partial}_S \tilde{f}) \} \quad \text{on } \omega_\alpha.$$

Putting (5.10) and (5.14) together, we finally have

$$(5.15) \quad f = \bar{\partial}_S \{ E(\tilde{f}) + \mathcal{E}(\bar{\partial}_S \tilde{f}) \} \quad \text{on } \omega_\alpha$$

as desired. This completes the proof of Theorem 5.1 when $f \in \Gamma_S^{p, q}$ and $q \leq n - 4$.

When $q = n - 3$, the above argument is valid except that we may not use Lemma 5.7 to show that $E_{234}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) = 0$. Instead, we use the following approximation argument used by Henkin in the hypersurface case (cf. [H, p. 90]).

Now clearly

$$E_{234}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f})(z) = \int_{\zeta \in S} K(\zeta, z) \wedge \bar{\partial}_S f(\zeta)$$

where $K(\zeta)$ is a $(n - p, 0)$ form with coefficients which are holomorphic in ζ provided $\zeta_1 \neq z_1, h(\zeta) \neq h(z)$ and $\partial \rho(z) / \partial z \cdot (\zeta - z) \neq 0$. Choose $\beta > \alpha$ so that $\text{supp } \bar{\partial}_S \tilde{f} \subset \{ \zeta \in S; \alpha \leq \text{Re } \zeta_1 \leq \beta \}$. Fix $z \in W_\alpha - S$ and set $t = r(z) \neq 0$. Let

$$B_\epsilon = \{ \zeta \in U; \rho(z) > -\epsilon, \alpha - \epsilon < \text{Re } \zeta_1 < \beta + \epsilon, \text{ and } -|t|/2 < r(\zeta) < |t|/2 \}.$$

We can choose $\epsilon > 0$ (depending on z) so that $K(\zeta, z)$ is holomorphic for $\zeta \in B_\epsilon$. Now recall that $\rho(\zeta) = \text{Re } \zeta_1 - v(\text{Im } \zeta_1, \zeta_2, \dots, \zeta_n)$ (with v strictly convex on U) and $r(\zeta) = \text{Re } h(\zeta)$ (with $h(\zeta)$ holomorphic on U). By considering the function $e^{h(\zeta)}$, it is easy to see that B_ϵ is relatively Runge in the open ball U . Therefore there is a

sequence $\{K_j(\zeta)\}$ of holomorphic $(n - p, 0)$ forms on U which converge uniformly on $\{\zeta \in S, \alpha \leq \text{Re } \zeta_1 \leq \beta\} \subset \subset B$ to $K(\zeta, z)$. Thus,

$$\begin{aligned} E_{234}([S]^{0,2} \wedge \bar{\partial}_S f)(z) &= \lim_{j \rightarrow \infty} \int_{\zeta \in S} K_j(\zeta) \wedge \bar{\partial}_S f(\zeta) \\ &= (-1)^{n-p+1} \lim_{j \rightarrow \infty} \int_{z \in S} \bar{\partial}_S K_j(\zeta) \wedge f(\zeta) = 0 \end{aligned}$$

as desired. This completes the proof of Theorem 5.1. Q.E.D.

REMARKS. (1) The solution, given in (5.15) to $\bar{\partial}_S u = f$, is valid for any smooth extension \tilde{f} with compact support in $U \cap S$. It is suggestive to try a particular nonsmooth extension, $\tilde{f} = \chi_{\omega_\alpha} f$, where χ_{ω_α} is the characteristic function on the set ω_α . Then $\bar{\partial}_S \tilde{f} = [\partial \omega_\alpha]^{0,3} \wedge f$ and the solution to $\bar{\partial}_S u = f$ on ω_α becomes

$$u = E(\chi_{\omega_\alpha} f) + \mathcal{E}([\partial \omega_\alpha]^{0,3} \wedge f).$$

The above solution only depends on the values of f on $\bar{\chi}_{\omega_\alpha}$ (and not on derivatives of f). This solution may be useful for certain L^p and continuity estimates. However, one must make sense out of the terms $E(\chi_{\omega_\alpha} f)$ and $\mathcal{E}([\partial \omega_\alpha]^{0,3} \wedge f)$ as smooth (or continuous) forms on ω_α . This, of course, is *not* implied by the boundary value theorems in [AB] which we used for the proofs of Lemmas 5.5 and 5.5'. We feel certain, however, that such an analysis can be carried through using similar techniques as in [AB].

(2) Looking back at equations (5.12) and (5.13), we see that the kernels E_{124} , E_{134} and E_{234} are the obstructions to the local solution to $\bar{\partial}_S$. The kernels E_{134} and E_{234} , however, only provide obstructions when they act on (p, q) forms with $q = 0$ or $n - 2$, by Lemma 5.7. (The kernel E_{124} is not an obstruction for an entirely different reason.) This is where the geometry on S (i.e., $\text{ex dim}(S) = 1$) really plays an important role. The geometric assumptions on S lead to the construction of u^3 and u^4 , which are holomorphic in both variables, which in turn is the key fact used in the proof of Lemma 5.7. If the geometry on S were different, then the kernels E_{134} and E_{234} (defined in any way that we can conceive of) would provide very nontrivial obstructions to solving $\bar{\partial}_S$.

(3) Looking over the work in [AB] and in the proof of Theorem 5.1, we see an inductive procedure forming to solve the tangential Cauchy-Riemann equations on higher codimension submanifolds. First, let us review what we have done. Let M be a strictly pseudoconvex hypersurface. Let $S \subset M$ be a submanifold defined by $S = \{z \in M; \text{Re } h(z) = 0\}$ where h is a holomorphic function. In [AB], the fundamental solution to $\bar{\partial}_S$ (the operator E) was derived from the fundamental solution to $\bar{\partial}_M$ on M . The content of Theorem 5.1 is that this fundamental solution can be modified so that we can locally solve the tangential Cauchy-Riemann equations on S .

Now suppose $Q \subset S$ is a generic submanifold defined by $Q = \{x \in S; \text{Re } k(x) = 0\}$ where k is a holomorphic function on some open set in \mathbb{C}^n . Following the same procedure as in [AB], the fundamental solution for $\bar{\partial}_Q$ can be derived from the fundamental solution for $\bar{\partial}_S$ on S . Then, using the same techniques as those in the

proof of Theorem 5.1, the fundamental solution to $\bar{\partial}_Q$ can be modified so that we can locally solve $\bar{\partial}_Q$, etc., etc. So, with the help of Theorem 3.4, one can prove the following analogue of Theorem 5.1 for higher codimension.

5.1'. THEOREM. *Suppose S is a real analytic regular submanifold of \mathbb{C}^n of real dimension $2n - k$. Suppose $\text{ex dim}(S) = 1$ and that S is strictly pseudoconvex at a point $x_0 \in S$. Then there exists a local neighborhood basis of open sets $\{\omega_\alpha\}$, $\alpha > 0$, about x_0 such that the following holds: For each $\alpha > 0$, there is an operator $K: \Gamma_S^{p,q}(\overline{\omega_\alpha}) \rightarrow \Gamma_S^{p,q-1}(\omega_\alpha)$ such that if $f \in \Gamma_S^{p,q-1}(\overline{\omega_\alpha})$, $1 \leq q \leq n - k - 1$, and $\bar{\partial}_S f = 0$ on ω_α , then $\bar{\partial}_S\{K(f)\} = f$ on ω_α .*

We feel no need to publish the proof of Theorem 5.1', because the details would be too repetitious of those in [AB] and in the proof of Theorem 5.1. Instead, we would like to raise the following interesting questions:

(1) Can we remove the real analytic assumption in Theorem 5.1 (and Theorem 5.1')?

The trick here would be to come up with a correct replacement for the function u^3 used in the proof of Theorem 5.1.

(2) For what class of submanifolds S with $\text{ex dim}(S) > 1$ can we locally solve the tangential Cauchy-Riemann equations with kernels?

We feel that there is some suitable answer to question (2), however the bidegree range in which we could solve the tangential Cauchy-Riemann equations would probably be more restrictive than the bidegree range handled in Theorem 5.1.

In Theorem 5.1, we were only interested in forms $f \in \Gamma_S^{p,q}(\omega_\alpha)$ where $1 \leq q \leq n - 3$. However, some interesting information can be obtained when $q = 0$. Our next theorem is our boundary value jump result for CR-functions. (A CR-function f on $\omega_\alpha \subset S$ satisfies $\bar{\partial}_S f = 0$ on ω_α ; see the definition of $\bar{\partial}_S$ in §4 and the remark after Definition 3.2.)

For V an open set in \mathbb{C}^n , let $\mathcal{O}(V)$ denote the space of holomorphic functions on V .

5.15'. THEOREM. *Let S and $x_0 \in S$ satisfy the hypothesis of Theorem 5.1. Locally there exists a real hypersurface M such that $S \subset M$ and S divides M into two pieces M^+ and M^- , and there exists a local neighborhood basis $\{W_\alpha\}$ for M about the point x_0 such that the following holds: For each W_α there exist open sets V_α^- and V_α^+ in \mathbb{C}^n which contain $W_\alpha \cap M^-$ and $W_\alpha \cap M^+$, respectively, such that $V_\alpha^+ \cap V_\alpha^- = \emptyset$, $W_\alpha \cap S \subset \overline{V_\alpha^+} \cap \overline{V_\alpha^-}$, and, moreover, given $f \in \Gamma^{p,o}(\overline{W_\alpha} \cap S)$ with $\bar{\partial}_S f = 0$ on $\overline{W_\alpha} \cap S$, there exists $F \in \mathcal{O}(V_\alpha^+ \cup V_\alpha^-)$ such that $F|_{M^- \cap W_\alpha}$ and $F|_{M^+ \cap W_\alpha}$ have continuous extensions to $W_\alpha \cap S$, denoted by F^- and F^+ , respectively; and, furthermore, $f = F^+ - F^-$ on $W_\alpha \cap S$.*

The theorem roughly states that a CR- (p, o) form on $\overline{W_\alpha} \cap S$ is the boundary value jump of a holomorphic (p, o) form defined on an open set, which only depends on $W_\alpha \cap S$.

PROOF OF THEOREM 5.15'. Let $M, M^+, M^-, W_\alpha, \omega_\alpha$, and U be defined as in the proof of Theorem 5.1. Let $\tilde{f} \in \mathcal{O}_S^{p,o}(S \cap U)$ be a smooth extension of the given

form f , with $\bar{\partial}_S \tilde{f} = 0$ on $\bar{\omega}_\alpha$. We first note that

$$E_{123}([S]^{0,2} \wedge \tilde{f})(z) = \int_{\zeta \in S} E_{123}(\zeta, z) \wedge \tilde{f}(\zeta) = 0$$

and

$$E_{1234}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f})(z) = \int_{\zeta \in S} E_{1234}(\zeta, z) \wedge \bar{\partial}_S \tilde{f}(\zeta) = 0$$

because of type considerations. For example, the degree of $E_{123}(\zeta, z) \wedge \tilde{f}(\zeta)$ in $d\bar{\zeta}$ is less than $n - 2$.

(5.9) and Lemma 5.7(a), therefore, yield

$$(5.16) \quad \tilde{f} = (E_{13}^+ - E_{13}^-)(\tilde{f}) + E(\bar{\partial}_S \tilde{f}) \quad \text{on } \omega_\alpha.$$

(5.12) and Lemma 5.7(b) yield

$$E_{123}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f}) = (E_{124} - E_{134})([S]^{0,2} \wedge \bar{\partial}_S \tilde{f})$$

on $W_\alpha - S$. Taking the boundary value jump of this equation across ω_α , and taking into account (5.8), we obtain

$$(5.17) \quad E(\bar{\partial}_S \tilde{f}) = (E_{123}^+ - E_{123}^-)(\bar{\partial}_S \tilde{f}) = (E_{134}^- - E_{134}^+)(\bar{\partial}_S \tilde{f}).$$

In view of (5.16) and (5.17), we finally have

$$(5.18) \quad f = (E_{13}^+ - E_{13}^-)(\tilde{f}) + (E_{134}^- - E_{134}^+)(\bar{\partial}_S \tilde{f}) \quad \text{on } \omega_\alpha.$$

Now, the functions u^1, u^2, u^3, u^4 are defined on $U \times U \subset \mathbf{C}^n \times \mathbf{C}^n$. Therefore, we may examine all the kernels defined above on $U \times U$, provided we avoid their singular sets. Define

$$\begin{aligned} F(z) &= E_{13}([S]^{0,2} \wedge \tilde{f})(z) - E_{134}([S]^{0,2} \wedge \bar{\partial}_S \tilde{f})(z) \\ &= \int_{\zeta \in S} E_{13}(\zeta, z) \wedge \tilde{f}(\zeta) - \int_{\zeta \in S} E_{134}(\zeta, z) \wedge \bar{\partial}_S \tilde{f}(\zeta) \end{aligned}$$

and let

$$\tilde{A}_i^z = \{ \zeta \in \bar{U}; u^i(\zeta, z) \cdot (\zeta - z) = 0 \}.$$

Clearly, $F(z)$ is smooth at a point $z \in U$ provided

$$(5.19) \quad \tilde{A}_1^z \cap S = \tilde{A}_3^z \cap S = \emptyset$$

and

$$(5.20) \quad \tilde{A}_4^z \cap \{ \text{supp } \bar{\partial}_S \tilde{f} \} \cap S = \emptyset.$$

We have already seen that if $z \in W_\alpha - S$ then

$$(5.21) \quad \tilde{A}_1^z \cap S = \tilde{A}_3^z \cap S = \emptyset.$$

Since each \tilde{A}_i^z is compact in \mathbf{C}^n for each z , it is clear that there is some neighborhood V_1 in \mathbf{C}^n containing $W_\alpha - S$ such that (5.21) remains true for $z \in V_1$.

Recall that $u^4(\zeta, z) \cdot (\zeta - z) = \zeta_1 - z_1$. Since $\bar{\partial}_S \tilde{f}(\zeta) = 0$ for $\text{Re } \zeta_1 \leq \alpha$, (5.20) clearly holds for z with $\text{Re } z_1 < \alpha$. Therefore if we let

$$\begin{aligned} V_\alpha^+ &= V_1 \cap \{z \in U; \text{Re } z_1 < \alpha\} \cap \{z \in U; r(z) > 0\}, \\ V_\alpha^- &= V_1 \cap \{z \in U; \text{Re } z_1 < \alpha\} \cap \{z \in U; r(z) < 0\}, \end{aligned}$$

then F is smooth on $V_\alpha^+ \cup V_\alpha^-$. Moreover, F is holomorphic on $V_\alpha^+ \cup V_\alpha^-$ because $u^1(\zeta, z)$, $u^3(\zeta, z)$, and $u^4(\zeta, z)$ are holomorphic functions in z for fixed ζ . Finally, (5.18) implies

$$f = F^+ - F^- \quad \text{on } \omega_\alpha. \quad \text{Q.E.D.}$$

EXAMPLE. Suppose $S = \{z \in \mathbf{C}^n; |z| = 1\} \cap \{z \in \mathbf{C}^n; \text{Re } z_n = 0\}$ is the equator of the unit sphere in \mathbf{C}^n . S satisfies the hypothesis of Theorem 5.1. We have $\rho(\zeta) = |\zeta|^2 - 1$, $u^1(\zeta, z) = \bar{\zeta}$, $u^2(\zeta, z) = \bar{z}$, and $u^3(\zeta, z) = (0, \dots, 1)$. Suppose $\omega_\alpha = \{z \in S; \text{Re } z_1 > 1 - \alpha\}$. Then we may take $u^4(\zeta, z) = (1, 0, \dots, 0)$. Theorem 5.1 says that we can solve $\bar{\partial}_S$ on ω_α on bidegrees (p, q) , $1 \leq q \leq n - 3$. Theorem 5.15' says that a CR-function on ω_α is the boundary value jump across ω_α of a holomorphic function defined on $V_\alpha = V_\alpha^+ \cup V_\alpha^-$ where we may take

$$\begin{aligned} V_\alpha^+ &= \{z \in \mathbf{C}^n; \text{Re } z_n > 0; \text{Re } z_1 > 1 - \alpha \text{ and} \\ &\quad |z_1|^2 + \dots + |z_{n-1}|^2 + (\text{Im } z_n)^2 < 1\}, \\ V_\alpha^- &= \{z \in \mathbf{C}^n; \text{Re } z_n < 0; \text{Re } z_1 > 1 - \alpha \text{ and} \\ &\quad |z_1|^2 + \dots + |z_{n-1}|^2 + (\text{Im } z_n)^2 < 1\}. \end{aligned}$$

The set V_α is constructed to be the largest open set such that (5.19) and (5.20) hold.

It is natural to ask whether or not one can arrange that one of F^+ or F^- vanish. That is, does there exist an open set V in \mathbf{C}^n with $\omega_\alpha \subset \bar{V} \cap S$ such that any CR-function on ω_α extends to a holomorphic function on V . For example, the answer is affirmative if $\text{ex dim}(S) = \text{codim}_{\mathbf{R}}(S)$; see for example [HW]. As the next theorem shows, however, the answer is negative when S is real analytic and $\text{ex dim}(S) < \text{codim}_{\mathbf{R}}(S)$. Thus, the boundary value jump result in the previous theorem is the best possible result.

5.22. THEOREM. *Suppose S is a real analytic, regular submanifold of real dimension $2n - k$ with $\text{ex dim}(S) < k = \text{codim}_{\mathbf{R}}(S)$. Suppose $x_0 \in S$ and ω is any sufficiently small open set in S containing x_0 . There does not exist any open set V in \mathbf{C}^n with the following properties:*

- (1) $\omega \subset \bar{V} \cap S$,
- (2) any CR-function on ω has a unique extension to a holomorphic function on V .

PROOF. Suppose $\text{ex dim}(S) = k - p$ with $p > 0$. By Theorem 3.4, there exist hypersurfaces N_j , $1 \leq j \leq p$, locally defined by $N_j = \{z \in \mathbf{C}^n; r_j(z) = 0\}$ where each $r_j: \mathbf{C}^n \rightarrow \mathbf{R}$ is real analytic and $dr_1 \wedge \dots \wedge dr_p \neq 0$, and there exists an open ball U in \mathbf{C}^n containing x_0 such that $S \cap U \subset N_1 \cap \dots \cap N_p$ and $\partial \bar{\partial} r_j = 0$ on U . Assume

that ω is any open set in S with $\omega \subset U \cap S$. We claim that $U \cap N_1 \cap \cdots \cap N_p$ is the intersection of domains of holomorphy. To this end we write

$$U \cap N_1 \cap \cdots \cap N_p = \bigcap_{t>0} V_t,$$

where

$$V_t = \{z \in U; -t < r_j(z) < t, 1 \leq j \leq p\}.$$

Since $\partial\bar{\partial}r_j = 0$ on U , for $1 \leq j \leq p$, each V_t is clearly a domain of holomorphy. So for each $t > 0$, there exists $f_t \in \mathcal{O}(V_t)$ such that f_t does not analytically continue past any part of ∂V_t (see Hörmander [Ho, Theorem 2.55]).

Now suppose, by contradiction, that there exists an open set V in \mathbf{C}^n satisfying properties (1) and (2). Choose $t > 0$ small enough so that $V \not\subset V_t$; then $f_t|_S$ is a CR-function on $S \cap U$ which extends to $\tilde{f}_t \in \mathcal{O}(V)$. Since this extension is unique, $f_t \in \mathcal{O}(V_t)$ has an extension $\tilde{f}_t \in \mathcal{O}(V)$, which is a contradiction. Q.E.D.

REMARKS. (1) The technique of writing a Levi flat surface, such as $N_1 \cap \cdots \cap N_p \cap U$, as the intersection of domains of holomorphy can also be found in Wells [W].

(2) Theorem 5.22 is *not* contradictory to Theorem 3.3 of Tomassini (used in the proof of Theorem 3.4) which states that any real analytic CR-function on a real analytic submanifold S is the restriction of a holomorphic function on a neighborhood of S . In Tomassini's theorem, the neighborhood is allowed to depend on the CR-function. The content of Theorem 5.22 is that there is no *fixed* neighborhood to which all CR-functions extend as holomorphic functions.

(3) As mentioned previously in the real analytic case, this result provides a converse to the following result (see Hunt and Wells [HW]): If S is a smooth regular submanifold of \mathbf{C}^n with $\text{ex dim}(S) = \text{codim}_{\mathbf{R}}(S)$, then S is locally extendible to an open set in \mathbf{C}^n . Thus, a real analytic, regular submanifold is extendible if and only if its excess dimension of its Levi algebra is maximal (i.e. $\text{ex dim} = \text{real codim}$).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109

Current address: Department of Mathematics, Texas A & M University, College Station, Texas 77843